Optimal Placement of Sensors and Actuators Using Measures of Modal Controllability and Observability in a Balanced Coordinate

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In this paper, a method for optimal placement of sensors and actuators is presented by using new measures of modal controllability and observability defined in a balanced coordinate system. The proposed new measures are shown to have a great advantage in practical use when they are used as criteria for selecting the locations of sensors and actuators, since the most controllable and observable locations can be obtained to be identical. In addition, they are more accurate than the measures of Hamdan and Nayfeh in that the effects of the eigenvector norm are considered into the magnitude of measures. In simulations, to verify the effectiveness of the proposed measures and optimal placement method, the closed-loop response of a simply supported flexible beam, in which the number and locations of actuators are determined by using the proposed measures and optimal placement method, has been examined and compared with the case of Hamdan and Nayfeh's measures.

Key Words : Measures of Modal Controllability and Observability, Balanced Coordinates, Optimal Placement of Sensors and Actuators, Large Space Structures

1. Introduction

In general, large space structures (LSS) such as a space station and the large solar arrays of a solar power station satellite have the characteristics of a flexible structure by the demands for light weight and large size. Hence, large space structures that are characterized by their inherent natures — infinite dimension, distributed parameter, low damping, and densely populated modes — and stringent performance requirements in space as well have been a topic of major concern (Balas, 1982 ; Hyland et al., 1993 ; Nurre et al., 1984 ; Yam et al., 1987). In large flexible structures, there are many vibration modes within the frequency band of disturbances and control bandwidth. Once they are disturbed, these modes are likely to remain excited for a long time because of their low natural frequency and small damping, which might hamper their missions in space. Therefore, to comply with the request of vibration suppression, the concept of actively controlled large flexible structures with sensors and actuators located on the structure is to be introduced.

Generally, a large number of sensors and actuators are required for the active vibration control of large flexible structures. Thus, the problem of choosing the appropriate number and locations of actuators and sensors is important, since an arbi-

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trary decision is expected to degrade the system performance and directly limit the range of practical applications (Junkins and Kim, 1993). In choosing the appropriate number and locations of sensors and actuators, our aim will be to excite the structure with minimum control effort for vibration suppression and also minimize the sensor signal power for the measurement of a given excitation of the structure. Here, the minimum control and measurement efforts are closely related with maximizing the degree of controllability and observability of a system, respectively. For instance, a poor system in which actuators are placed on or near the nodes(or node line) of vibration mode requires an excessively large control force at best, or uncontrollable at worst. Consequently, the optimal placement of actuators and sensors to maximize the degrees of controllability and observability can improve the control and estimation performance of a closed-loop system.

A large number of strategies for optimal placement of sensors and actuators have been developed based on the concept of controllability and observability, which are derived from minimum energy consideration. However, the conventional controllability/observability test, which presents binary information only: controllable (observable) or uncontrollable (unobservable), does not provide a graduated measure of how controllable or observable a system is. Hence, special interest is given to the development of explicit relationship between the system's controllability(or observability) and vibration modes. From this consideration, there have been several research contributions of Longman et al. (1982), Moore (1981), and Hamdan and Nayfeh (1989a), which are all focused on establishing the quantitative measures of controllability and observability. Among them, Hamdan and Nayfeh proposed the measures of modal controllability and observability by introducing the generalized angles between two vector spaces: the left eigenvectors and the column vectors of input influence matrix for modal controllability measure; the right eigenvectors and the row vectors of output measurement matrix for modal observability measure. Hamdan and Nayfeh's measures are most attractive and provide us with explicit information on each mode controllability and observability.

2. Modeling of Large Flexible Structures

However, there are two important issues to be solved when these measures are used as a criterion for the selection of optimal locations of sensors and actuators. First, in Hamdan and Nayfeh's measures, only the generalized angles between the left eigenvectors(right eigenvectors) and the input vectors(output vectors) are focused on and the norms of two vectors are neglected, even though there evidently exist the norms of the left eigenvectors and the input vectors. However, the neglected norms must be included for accurate measures. For this reason, Choi et al. (1995) proposed a novel measure which includes the norm of input vectors. However, it is difficult to deal with the norms of the eigenvectors since they are scaled arbitrarily. Second, even if the most controllable set of actuator locations is chosen, we can not assure that the collocated set of sensor locations is the most observable one. Namely, it is difficult to achieve the equally controllable and observable location set of actuators and sensors by using Hamdan and Nayfeh's measures.

In this paper, it is shown that these two issues can be solved by introducing the balanced coordinate transformation, based on the facts that the measures of controllability and observability are variant under a coordinates transformation (Aquirre, 1995) and, furthermore, a certain mode in balanced coordinate system is equally controllable and observable. Hence, by using the novel measures defined in the balanced coordinate system, we can obtain the optimal location set of sensors and actuators which equally maximizes controllability and observability of a system and need not take any pain in considering the norms of eigenvectors into the measures. In addition, the additional measures are introduced for the simple and explicit optimal placement method. To evaluate the effectiveness of the proposed measures and optimal placement method, the closed-loop responses of a simply supported uniform flexible beam, in which the number and locations of actuators are determined by using the proposed measures and optimal placement method and the controller is designed by using the left eigenstructure assignment scheme (Choi et al., 1995), are investigated and compared with the case of Hamdan and Nayfeh's measures.

By using an approximation technique such as the finite element method, large flexible structures which are inherently infinite dimensional and distributed parameter system can be modeled as a lumped parameter and finite dimensional system. To define the model of large flexible structures, we consider the second order mechanical vibrating systems as follows:

$$M\ddot{\eta}(t)+D\dot{\eta}(t)+K\eta(t)=Fu(t),\ \eta(t_0)=\eta_0 \quad (1)$$

In Eq. (1), $\eta(t)$ denotes a displacement vector and $u(t)$ is control input vector via point forcing actuators. M, D, and K represent the lumped mass, damping, and stiffness matrices, respectively, and F is the input force influence matrix that indicates the way the input force acts on a flexible structure. Here, we assume that identical actuators having equal capability are used, hence only the information about actuator's locations is contained in F.

If the state vector is defined as $x=\{\dot{\eta} \quad \eta\}^T$, the second order systems in Eq. (1) is described in the state space representation of a linear, time-invariant system as follows:

$$\begin{aligned}\dot{x}(t)&=Ax(t)+Bu(t)\\&=\begin{bmatrix}0 & I\\ -M^{-1}K & -M^{-1}D\end{bmatrix}x(t)+\begin{bmatrix}0\\ M^{-1}F\end{bmatrix}u(t) \quad (2)\\ y(t)&=Cx(t)\end{aligned}$$

where $x\in R^n$, $u\in R^m$, and $y\in R^l$ are the state, control input, and output measurement vectors, respectively. A, B, and C are, respectively, constant matrices of system, input influence, and output measurement with appropriate dimension. It is assumed that (B, A) is controllable, (C, A) is observable, and B is an $n\times m$ matrix with a full column rank. In addition, the collocated system in which sensors and actuators are placed in the same direction and position of the structure is assumed, then $C=B^T$ exists in Eq. (2). It is well known that non-collocated system becomes a non-minimum phased system, which is difficult to control, and the collocation of sensors and actuators is preferable from the robustness viewpoint as well.

Then, the system in Eq. (2) can be described by using modal decomposition ($A=\Phi\Lambda\Psi^T$ as follows:

$$\dot{x}(t)=(\Phi\Lambda\Psi^T)x(t)+Bu(t) \quad (3)$$

where $\Lambda\in R^{n\times n}$ is the diagonal matrix of eigenvalues, and $\Phi\in R^{n\times n}$ and $\Psi\in R^{n\times n}$ are, respectively, the right and left modal matrices whose columns consist of the right and left eigenvectors of A. These eigenvectors can be determined by the solution of the standard eigenvalue problem (Chen, 1979). If the initial value is assumed to be zero, then the output response of system in Eq. (3) can be obtained as follows (Kailath, 1980):

$$y=C\Phi\int_0^{t_f}e^{\Lambda(t-\tau)}\{\Psi^TB\}u(\tau)\,d\tau,\ y(0)=0 \quad (4)$$

In Eq. (4), the matrix term Ψ^TB represents the channels from the control input to the system's modes, and the matrix $C\Phi$ represents the channels from the system's modes to the output. These two matrix terms are closely related with system's controllability and observability, which will be carefully discussed in the next section.

3. Measures of Modal Controllability and Observability in Balanced Coordinates

In this section, the novel measures of modal controllability and observability that are defined in the balanced coordinate system are presented. These measures are more preferable than ones defined in the original coordinate system from the viewpoint that they are accurate and practically useful for the optimal placement of sensors and actuators.

From Eq. (4), it can be easily seen that the matrix Ψ^TB has an important information about the way the control input $u(t)$ has influence on the modes of system and can be described as follows:

$$\Psi^T B=\begin{bmatrix} \psi_1^T b_1 & \cdots & \psi_1^T b_m \\ \vdots & \ddots \quad \psi_i^T b_j \quad \ddots & \vdots \\ \psi_n^T b_1 & \cdots & \psi_n^T b_m \end{bmatrix} \tag{5}$$

whose ij-th entry, inner product of the i-th left eigenvector and the j-th input vector, represents the degree of the i-th mode's controllability from the j-th control input and is described as follows :

$$| \psi_i^T \cdot b_j \| = \| \psi_i \| \| b_j \| \cos \theta_{ij}, \quad i=1, 2, \cdots, n, \; j=1, 2, \cdots, m \tag{6}$$

where θ_{ij} denotes the angle between the input vector b_j and the left eigenvector ψ_i. In Eq. (6), the magnitude of $\psi_i^T \cdot b_j$ can be taken as an indication for the modal controllability of the i-th mode from the j-th input.

In addition, the matrix $C\Phi$ in Eq. (4) indicates how much the individual modes participate in each outputs. The matrix $C\Phi$ and its ki-th element can also be described as follows :

$$C\Phi=\begin{bmatrix} c_1\phi_1 & \cdots & c_1\phi_n \\ \vdots & \ddots \quad c_k\phi_i \quad \ddots & \vdots \\ c_l\phi_1 & \cdots & c_l\phi_n \end{bmatrix} \tag{7}$$

$$| c_k \cdot \phi_i \| = \| c_k \| \| \phi_i \| \cos \delta_{ki}, \quad i=1, 2, \cdots, n, \; k=1, 2, \cdots, l$$

where δ_{ki} denotes the angle between the output vector c_k and the right eigenvector ϕ_i. The magnitude of $c_k \cdot \phi_i$ can be taken as an indication for the modal observability of the i-th mode to the k-th output. Then, we investigate the possibility that the magnitude of two inner products in Eq. (6) and Eq. (7) defined in the balanced coordinate system can be new measures of modal controllability and observability, respectively, and the usefulness when they are used as criteria for the optimal placement of sensors and actuators.

From Eq. (6), Hamdan and Nayfeh make much of the angle between two vectors, regarding that the dependence of $| \psi_i^T \cdot b_j |$ on $\cos \theta_{ij}$ is dominant. By introducing a geometrical interpretation of the Popov, Belevitch, and Hautus (PBH) eigenvector test, they proposed the measures of modal controllability and observability as follows (Hamdan and Nayfeh, 1989a):

$$\nu_{ij} = \cos \theta_{ij} = \frac{| \psi_i^T \cdot b_j |}{\| \psi_i \| \| b_j \|}, \quad \mu_{ki} = \cos \delta_{ki} = \frac{| c_k \cdot \phi_i |}{\| c_k \| \| \phi_i \|} \tag{8}$$

They concentrate on the fact that $\cos \theta_{ij}$ and $\cos \delta_{ki}$ is dominant in $| \psi_i^T \cdot b_j |$ and $| c_k \cdot \phi_i |$, respectively, when both angles θ_{ij} and δ_{ki} are either right angles or very near to it. In addition, Choi, et al. proposed a new version of Eq. (8) by considering the norms of input vector b_j and output vector c_k into the above measures as follows (Choi et al., 1995):

$$\nu_{ij} = \| b_j \| \cos \theta_{ij} = \frac{| \psi_i^T \cdot b_j |}{\| \psi_i \|}, \quad \mu_{ki} = \| c_k \| \cos \delta_{ki} = \frac{| c_k \cdot \phi_i |}{\| \phi_i \|} \tag{9}$$

It is obvious that these measures are all attractive in that they can represent the system's modal controllability and observability explicitly. However, there remains two problems in applying these measures as criteria for the optimal placement of sensors and actuators in control of large flexible structure.

First is that the norms of the eigenvectors, although they are arbitrarily scaled, are not considered in these measures and only the directions of eigenvectors are considered. For further examination, the relationship between the gross measure of modal controllability/observability and the residue matrix is investigated. Hamdan and Nayfeh suggest the gross measures of modal controllability and observability of a given mode, which represent a projected magnitude of each eigenvectors into the input and output spaces. Here, the i-th gross measures of modal controllability (σ_i) and observability (ω_i) are defined, respectively, as follows (Hamdan and Nayfeh, 1989a):

$$\sigma_i = \| f_i \|, \quad f_i = \frac{\psi_i^T B}{\| \psi_i \|} = [\| b_1 \| \cos \theta_{i1} \; \| b_2 \| \cos \theta_{i2} \cdots \| b_m \| \cos \theta_{im}] \tag{10}$$

$$\omega_i = \| h_i \|,$$
$$h_i = \frac{C\phi_i}{\| \phi_i \|} = [\| c_1 \| \cos\delta_{1i} \; \| c_2 \| \cos\delta_{2i} \; \cdots \; \| c_l \| \cos\delta_{li}]^T \tag{11}$$

Then, the transfer function $G(s)=C(sI-A)^{-1}B$ of Eq. (2) can be expanded as follows :

$$G(s) = \sum_{i=1}^{n} \frac{R_i}{s-\lambda_i} \tag{12}$$

where the residue matrix R_i is

$$R_i = C\phi_i\psi_i^T B, \; i=1, 2, \cdots, n \tag{13}$$

The residue matrix $R_i (i=1, 2, \cdots, n)$ is related to some properties of the state space and frequency-response representation, and invariant under any coordinate transformation. From Eq. (13), by taking norms on both sides, the relationship between the norm of residue matrix and the gross measure of controllability/observability can be described as follows (Hamdan and Nayfeh, 1989a ; Linder et al., 1989):

$$\| R_i \| = \| \phi_i \| \| \psi_i \| \| h_i \| \| f_i \|, \; i=1, 2, \cdots, n \tag{14}$$

In Eq. (14), it can be seen that the norm of R_i, which indicates the contribution of the i-th mode to the system's entire input-output relations, depends on the joint gross measure of modal controllability and observability ($\| h_i \| \| f_i \|$). Note that there exists a difference between the norm of the residue matrix R_i and the joint gross measures of modal controllability and observability, which corresponds to $\| \phi_i \| \| \psi_i \|$. Besides, $\| \phi_i \| \| \psi_i \|$ has some values that are greater than unity if the i-th mode is not perfectly well-conditioned. Hence, for accurate measures, it is desirable to include the norms of the left and right eigenvectors, then the residue matrix R_i is exactly identical with the joint gross measures of modal controllability and observability. However, it is difficult to explicitly decide the norms of the left and right eigenvectors, respectively, due to their inherent arbitrary scaling.

Second problem is that, if actuators and sensors are collocated, the selected locations that are the most controllable for actuators do not always guarantee the most observable ones for sensors. Such an undesirable location set can not improve the control efficiency, hence the equally controllable and observable locations are more preferable. However, by using the measures of Hamdan and Nayfeh, it is difficult or complicated to achieve the equally controllable and observable location set of sensors and actuators.

From these two considerations, our attention is paid to a coordinate transformation such that the measures of modal controllability and observability become equal and the norms of the left and right eigenvectors are identical for a certain mode ; this unique set of coordinates is called internally balanced. For the purpose of model reduction, Moore (1981) introduced an internally balanced system by using the singular values to define the measures of nearness to rank deficiency of the controllability and observability grammians.

Let us transform the original system in Eq. (2) into a balanced coordinate system, which has a parity symmetric matrix $\bar{A}$ as follows (Moore, 1981):

$$\begin{aligned} \dot{z}(t) &= \bar{A}z(t) + \bar{B}u(t) \\ y(t) &= \bar{C}z(t) \end{aligned} \tag{15}$$

In Eq. (15), there exists an interesting symmetric property of eigenvectors as follows :

$$\bar{\phi}_i = E\bar{\psi}_i \tag{16}$$

where E is a diagonal matrix with entries ± 1.

Using the Eq. (16), Hamdan and Nayfeh show that the following relationship between the measures of modal controllability and observability exists in the balanced coordinate system (Hamdan and Nayfeh, 1989b):

$$\| (\bar{\psi}_i^T \bar{B})^T \| = \| \bar{C}\bar{\phi}_i \| \tag{17}$$

which means that balancing makes the modes equally controllable and observable. Actually, every direction in the state space is equally controllable and observable. Hence, we proposed the two inner products in Eq. (6) and Eq. (7) as new measures of modal controllability and observability when they are defined in the balanced coordinate system of Eq. (15) as follows :

$$\begin{aligned} &\bar{\nu}_{ij}=|\bar{\psi}_i^T\cdot\bar{b}_j|=\|\bar{\psi}_i\|\|\bar{b}_j\|\cos\bar{\theta}_{ij}, \\ &i=1,2,\cdots n,\ j=1,2,\cdots,m \\ &\bar{\mu}_{ki}=|\bar{c}_k\cdot\bar{\phi}_i|=\|\bar{c}_k\|\|\bar{\phi}_i\|\cos\bar{\delta}_{ki}, \\ &i=1,2,\cdots n,\ k=1,2,\cdots,l \end{aligned} \tag{18}$$

where always exists $\bar{\nu}_{ij}=\bar{\mu}_{ki}$. Based on these measures, the selection method for the optimal number and location of sensors and actuators is developed in the next section.

4. Optimal Placement of Sensors and Actuators

In this and next sections, the optimal placement of sensors and actuators which employs the measures of modal controllability and observability in balanced coordinates in Eq. (18) as its criterion is presented. In addition, for a simple and clear method, the additional measures are introduced in this section.

In general, the determination of the optimal number and locations of sensors and actuators results in the complex optimization problems such as integer programming, linear programming, and non-linear programming (Maghami and Joshi, 1993). As the number of possible combinations for optimal locations of sensors and actuators increases, an exhaustive search for a global optimum is computationally infeasible and gradient based techniques typically produce local minima. For large flexible structures having a large number of sensors and actuators, the optimization procedure will become even more complicated, with much more increased local minima. Hence, the development of a systematic and computationally feasible search strategy to actually solve for the optimal set of sensors and actuators is still required in this problem.

From here, we examine the optimal placement method for actuators only, because sensors can be placed in the same procedure when the measures in balanced coordinates are used as its criterion as mentioned previously. This means that once the most controllable location set of actuators is selected, then the equally observable location set of sensors can be achieved automatically in the collocated system.

First, note that the measure matrix $\bar{\Psi}^T\bar{B}$ can be written as follows:

$$\bar{\Psi}^T\bar{B}=\begin{bmatrix} \bar{\psi}_1^T\bar{b}_1 & \bar{\psi}_1^T\bar{b}_2 & \cdots & \bar{\psi}_1^T\bar{b}_m \\ \bar{\psi}_2^T\bar{b}_1 & \bar{\psi}_2^T\bar{b}_2 & \cdots & \bar{\psi}_2^T\bar{b}_m \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\psi}_n^T\bar{b}_1 & \bar{\psi}_n\bar{b}_2 & \cdots & \bar{\psi}_n^T\bar{b}_m \end{bmatrix} \tag{19}$$

where each entries represent the ij-th measure of modal controllability of the i-th mode from the j-th actuator's input as defined in Eq. (18). However, when the problem of optimal placement of actuators is considered, it is difficult to draw any explicit indication about the variation of the entire system's controllability according to the possible candidate set of actuator locations from these measures themselves. Hence, the additional measures are introduced to derive the more evident indication about how the controllability of the entire system varies with respect to the number and locations of actuators.

In Eq. (19), the norm of the i-th row vector indicates the degree of the i-th mode's controllability from all inputs that is induced by a specified set(number and locations) of actuators, and is defined as follows:

$$\begin{aligned} &\bar{\sigma}_i=\|\bar{f}_i\|,\ \bar{f}_i=\bar{\psi}_i^T\bar{B}=[\bar{\psi}_i^T\bar{b}_1\ \ \bar{\psi}_i^T\bar{b}_2\ \cdots\ \bar{\psi}_i^T\bar{b}_m], \\ &i=1,2,\cdots,n \end{aligned} \tag{20}$$

where $\bar{\sigma}_i$ represents the i-th mode's gross measure of modal controllability. $\bar{\sigma}_i$ can be physically interpreted as representing the power injected by all actuators into the i-th mode and the relative degree of the i-th mode's contribution in the system's entire input-output relations as well. Hence, the gross measures of modal controllability of each modes are given as follows:

$$\bar{F}=[\bar{\sigma}_1\ \bar{\sigma}_2\ \cdots\ \bar{\sigma}_n]^T=[\|\bar{f}_1\|\ \|\bar{f}_2\|\ \cdots\ \|\bar{f}_n\|]^T \tag{21}$$

From Eqs. (20) and (21), a measure that represents the total degree of controllability of the entire system can be defined as follows:

$$\bar{\rho}=\|\bar{F}\|=\sum_{i=1}^{n}\bar{\sigma}_i \tag{22}$$

where $\bar{\rho}$ gives us a direct indication for the selection of the appropriate number of actuators and can be used as an important index for the optimal placement method.

Next, the other additional measure that gives a direct indication for the selection of actuator locations is examined. When we consider the norm of the j-th column vector in Eq. (19), it can be seen that this norm represents the j-th actuator's contribution to the controllability of all modes, and is defined as follows:

$$\bar{\xi}_j = \| \bar{q}_j \|,\ \bar{q}_j = \bar{\Psi}^T \bar{b}_j = [\bar{\psi}_1^T\ \ \bar{\psi}_2^T \bar{b}_j\ \ \bar{\psi}_n^T \bar{b}_j]^T, \quad j = 1, 2, \cdots, m \tag{23}$$

where $\bar{\xi}_j$ represents the gross measure of input controllability that indicates the degree of controllability of all n modes with respect to the j-th actuator. Then, for the whole set of actuator locations, the following vector that consists of the measure $\bar{\xi}_j$ can be obtained:

$$\bar{Q} = [\bar{\xi}_1\ \bar{\xi}_2 \cdots \bar{\xi}_m] = [\| \bar{q}_1 \|\ \| \bar{q}_2 \| \cdots \| \bar{q}_m \|] \tag{24}$$

In vector $\bar{Q}$, the largest entry indicates by the entry's position itself that the corresponding actuator location is the most controllable one in the whole set of locations.

By using the additional measures in Eqs. (20), (22) and (23) that are all derived from the novel measures of modal controllability in balanced coordinates, the optimal number and locations of actuators can be determined in the simple and evident manner. Then, the optimal number and locations of sensors are the same as those of actuators by the helpful properties of balanced coordinate system as examined in the previous section. The detailed calculation procedures are presented in the next section through a numerical example.

5. A Numerical Example

In numerical experiments, the proposed method for the optimal placement of sensors and actuators is applied to a simply supported flexible beam that is modeled to have six vibration modes. As illustrated in Fig. 1, a flexible beam model in which lumped unit masses are placed at uniformly distributed six nodes and both damping and stiffness effects are presented by continuous beam structure is used. In simulations, the beam has the length(L) of 1 m and the bending stiffness(EI)

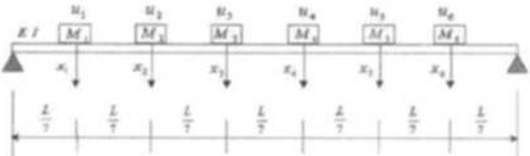


Fig. 1 A simply supported beam model

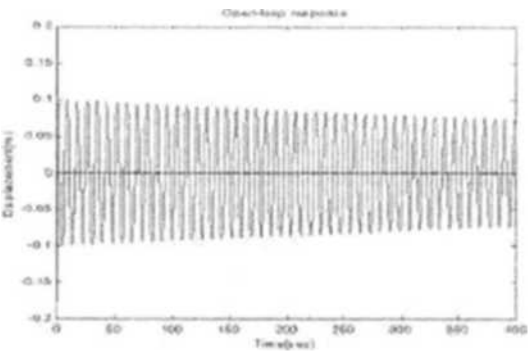


Fig. 2 Response of the open-loop system

of 0.001 N/m, and also the proportional damping $D = 0.001M + 0.001$ K is assumed. By first assuming that six independent actuators are placed at each nodes, the state space representation of the flexible beam model under consideration is described in Appendix A.1. Then, the eigenvalues of the open-loop beam are as follows:

$$\Lambda_{open} = \begin{Bmatrix} -0.0682 \pm 11.6332i & -0.0399 \pm 8.8776i & -0.0178 \pm 5.8829i \\ -0.0061 \pm 3.3458i & -0.0016 \pm 1.4913i & -0.0006 \pm 0.3730i \end{Bmatrix}$$

It is obvious from the open-loop response depicted in Fig. 2 that the beam model has the characteristic of a flexible structure.

From now on, to evaluate the effectiveness of the proposed measures, two cases of the optimal placement of actuators are carried out on the previous flexible beam model and compared each other: One uses the measures proposed in this study as its criteria, and the other uses the measures proposed by Hamdan and Nayfeh (1989a). After the locations of actuators for two cases are determined, their closed-loop responses are compared to investigate which location set is more controllable than the other.

First, as for the case of the proposed measures, the measure matrix $\bar{\Psi}^T \bar{B}$ in Eq. (19) and the additional measures in Section 4 are obtained in Appendix A.2. In Appendix A.2, each entry of vector $\bar{F}$, which is composed of the norm of each row vectors of $\bar{\Psi}^T\bar{B}$, represents the gross measure of modal controllability of a corresponding mode as in Eq. (20). In addition, each entry of vector $\bar{Q}$, which is composed of the norm of each column vectors of $\bar{\Psi}^T\bar{B}$, represents the gross measure of input controllability as in Eq. (23). Then, this vector gives an indication about which actuator's location has the largest influence on the whole modes and can be used as a direct index for the selection of actuator locations. It is readily seen that the 2nd and 5th actuator locations (the under-lined entries) represent the most controllable ones among six candidate locations. Finally, $\bar{\rho}$ is the norm of vector $\bar{F}$ representing the measure of total controllability of all modes with respect to the current configuration set of six actuators and can be used for the selection of number of actuators.

Here, all the possible candidate configurations of actuator locations according to its number are summarized in Table 1, which gives the explicit indication about how to select the optimal number and locations of actuators. In Table 1, the most controllable location set for the corresponding number (m) is determined by examining the elements of vector $\bar{Q}$, and then the measure of total controllability ($\bar{\rho}$) for each configurations (number and locations) set is calculated from Eq. (22). From this, for the selection of the number of actuators, there has to be a compromise between the control efficiency and economical aspects by taking into account of the improvement of the measure of total controllability ($\bar{\rho}_m-\bar{\rho}_{m-1}$). In this example, we can easily select the 2nd configuration set in Table 1 in which two actuators are located at the 2nd and 5th nodes, because the improvement of $\bar{\rho}$ is remarkably large in this configuration set as seen from $\bar{\rho}_2-\bar{\rho}_1$. Note that the same number and location set for sensors can be automatically achieved to be the most observable ones by the help of new measures in balanced coordinates. However, a complex search algorithm is still needed when the larger number of actuators and the more complicated configuration set are involved.

Next, to determine the actuator's location set for the case of the Hamdan and Nayfeh's measures, Ψ^TB in the original system is presented in Appendix A.3 obtained by using the measure of modal controllability in Eq. (8). From the vector Q in Appendix A.3, the two actuator locations, the 3rd and 4th nodes, can be selected for the most controllable ones. However, the most observable locations for two sensors, the 1st and 6th nodes, are not coincided with those of actuators as can be seen from Appendix A.4, where the vectors P and G, respectively, correspond to the vectors Q and F in the case of controllability measures in Appendix A.3. That is, if the sensors are collocated with the actuators, then the most observable locations for sensors can not be achieved.

From here, the closed-loop responses for the previously described two cases of actuator location set are compared to investigate which location set is more controllable. For the vibration suppression of the flexible beam, a state feedback control using the left eigenstructure assignment (Choi et al., 1995) is applied. Let the desired eigenvalues of the closed-loop system be assigned as follows:

$$\Lambda_{closed}=[-1.0\ -1.1\ -1.3\ -1.4\ -1.7\ -2.0\ -2.1\ -2.3\ -2.5\ -2.6\ -2.8\ -4.0]$$

Then, the control gain matrix and its norm can be obtained in both cases as follows:

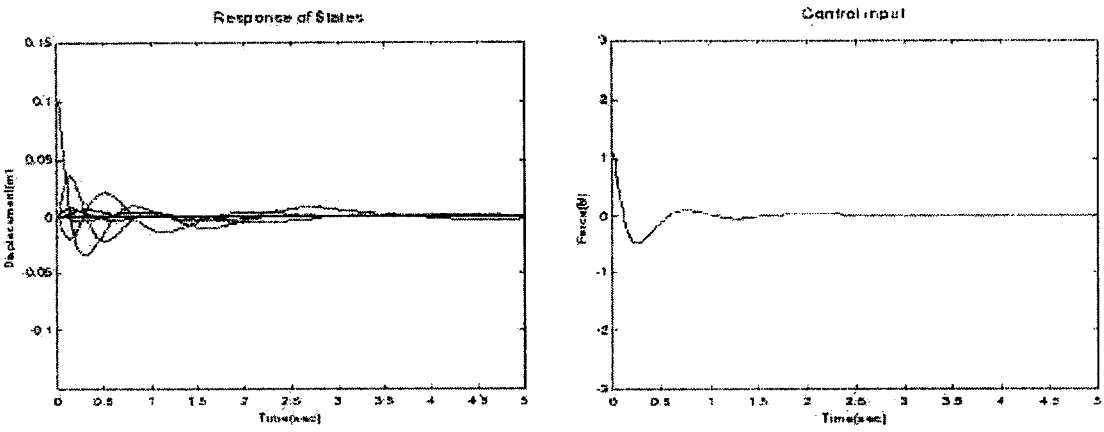
CASE 1: Actuator locations at nodes 2 and 5

$$K=\begin{bmatrix} -0.0145 & 0.0538 & -9.8216 & 22.0453 & -30.6325 & -108.8997 & -0.1411 & 0.0316 & -0.2769 & 6.1775 & 10.9546 & 19.1708 \\ -0.0145 & -0.0538 & -9.8216 & -22.0453 & -30.6325 & 108.8997 & -0.1411 & -0.0316 & -0.2769 & -6.1775 & 10.9546 & -19.1708 \end{bmatrix}$$

$$\| K \|_{fro}=166.7677$$

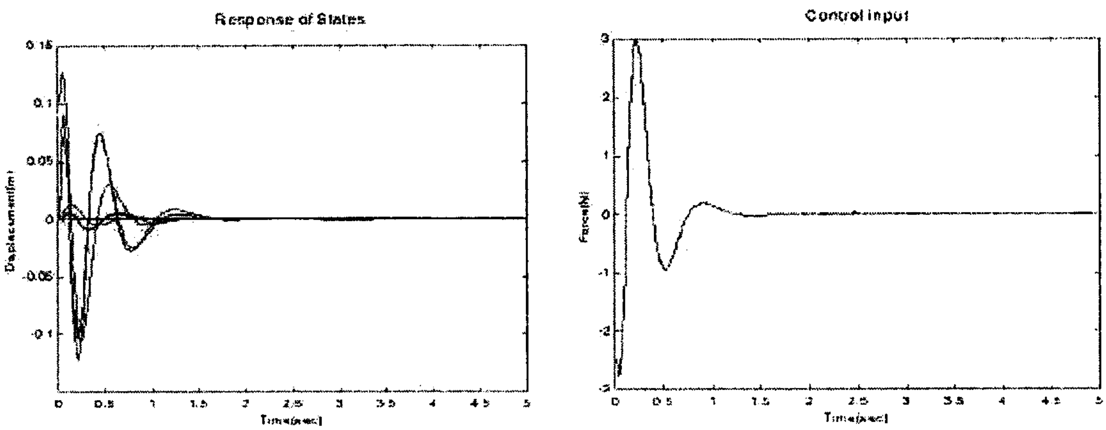
Table 1 Candidate configurations (number and locations) of actuators

number (m)	1	2	3	4	5	6
location (node)	2 or 5	2, 5	1, 2, 5 or 2, 5, 6	1, 2, 5, 6	1, 2, 3, 5, 6 or 1, 2, 4, 5, 6	1, 2, 3, 4, 5, 6
$\bar{\rho}$	22.4303	37.7212	38.5983	44.4229	49.1209	53.4072
$\bar{\rho}_m-\bar{\rho}_{m-1}$		15.2907	0.8771	5.8296	4.6980	4.2863



(a) (b)

Fig. 3 Responses of the states and control inputs for CASE 1(nodes 2 and 5)



(a) (b)

Fig. 4 Responses of the states and control inputs for CASE 2(nodes 3 and 4)

CASE 2 : Actuator locations at nodes 3 and 4

$$K=\begin{bmatrix} -0.0116 & 0.1208 & 5.4506 & 12.2342 & 68.8305 & 87.3308 & -0.1132 & 0.0711 & 0.1537 & 3.4282 & -24.6148 & -15.3738 \\ -0.0116 & -0.1208 & 5.4506 & -12.2342 & 68.8305 & -87.3308 & -0.1132 & -0.0711 & 0.1537 & -3.4282 & -24.6148 & 15.3738 \end{bmatrix}$$

$$\| K \|_{fro}=163.6932$$

where $\|\cdot\|_{fro}$ represents the Frobenious norm of $[\cdot]$. Note that these two cases are compared in the conditions of the same closed-loop eigenvalues and the nearly same norm of control gan matrices.

Then, the initial value responses of the closed-loop system and their control input history for both cases can be obtained as shown in Figs. 3 and 4. In Fig. 3, it is shown that the vibration suppression in CASE 1 is well performed but needs more convergence time than CASE 2. However, in contrast to CASE 1, CASE 2 undergoes more oscillated transition to fast convergence and needs more control energy than CASE 1 as seen from Fig. 4. From these results, it can be said that CASE 1 requires much less control energy than CASE 2 to achieve the nearly same control performance of vibration suppression. That is, the actuator location set in CASE 1 (nodes 2 and 5), which is selected by using the proposed measures as criteria, is shown to be more controllable than that of CASE 2 (nodes 3 and 4) that is selected by using the measures of Hamdan and Nayfeh.

6. Conclusions

This paper presents new measures of modal controllability and observability defined in the balanced coordinate system, which are more accurate and practically useful than the conventional measures of Hamdan and Nayfeh. By using these measures, we can obtain the optimal location set of sensors and actuators which equally maximizes controllability and observability of a system and need not take any pain in considering the norms of eigenvectors into the measures, because a balanced coordinate system has equally controllable and observable modes and the left and right eigenvectors of a certain mode have the same magnitude of norms. In addition, by the help of the additional measures, more simple and explicit optimal placement method is presented. The effectiveness of the proposed measures and optimal placement method to improve the control efficiency for the control system of large flexible structures has been verified by numerical simulations.

Appendix

A.1 The state space model of a flexible beam

$$\dot{x}(t)=Ax(t)+Bu(t)$$

$$= \left[\begin{array}{cccccc|cccccc} \multicolumn{6}{c|}{[0]_{6\times6}} & \multicolumn{6}{c}{I_{6\times6}} \\ \hline -33.8782 & 32.6127 & -14.2526 & 3.8177 & -1.0180 & 0.2545 & -0.0349 & 0.0326 & -0.0143 & 0.0038 & -0.0010 & 0.0003 \\ 32.6127 & -48.1308 & 36.4303 & -15.2706 & 4.0722 & -1.0180 & 0.0326 & -0.0491 & 0.0364 & -0.0153 & 0.0041 & -0.0010 \\ -14.2526 & 36.4303 & -49.1488 & 36.6849 & -15.2706 & 3.8177 & -0.0143 & 0.0364 & -0.0501 & 0.0367 & -0.0153 & 0.0038 \\ 3.8177 & -15.2706 & 36.6849 & -49.1488 & 36.4303 & -14.2526 & 0.0038 & -0.0153 & 0.0367 & -0.0501 & 0.0364 & -0.0143 \\ -1.0180 & 4.0722 & -15.2706 & 36.4303 & -48.1308 & 32.6127 & -0.0010 & 0.0041 & -0.0153 & 0.0364 & -0.0491 & 0.0326 \\ 0.2545 & -1.0180 & 3.8177 & -14.2526 & 32.6127 & -33.8782 & 0.0003 & -0.0010 & 0.0038 & -0.0143 & 0.0326 & -0.0349 \end{array}\right] x(t)$$

$$+\left[\begin{array}{c} [0]_{6\times6} \\ \hline I_{6\times6} \end{array}\right] u(t)$$

$$y(t)=Cx(t)=B^{T}x(t)$$

A.2 The proposed controllability measures

$$\bar{F}=\begin{bmatrix} 18.0958 \\ 18.0958 \\ 9.2374 \\ 9.2374 \\ 10.5466 \\ 10.5466 \\ 12.8534 \\ 12.8534 \\ 16.5641 \\ 16.5641 \\ 21.5072 \\ 21.5072 \end{bmatrix} \qquad \bar{\Psi}^{T}\bar{B}=\begin{bmatrix} 4.1968 & 7.5624 & 9.4301 & 9.4301 & 7.5624 & 4.1968 \\ 4.1968 & 7.5624 & 9.4301 & 9.4301 & 7.5624 & 4.1968 \\ 2.1423 & 3.8604 & 4.8138 & 4.8138 & 3.8604 & 2.1423 \\ 2.1423 & 3.8604 & 4.8138 & 4.8138 & 3.8604 & 2.1423 \\ 4.4075 & 5.4961 & 2.4460 & 2.4460 & 5.4961 & 4.4075 \\ 4.4075 & 5.4961 & 2.4460 & 2.4460 & 5.4961 & 4.4075 \\ 6.6982 & 2.9810 & 5.3715 & 5.3715 & 2.9810 & 6.6982 \\ 6.6982 & 2.9810 & 5.3715 & 5.3715 & 2.9810 & 6.6982 \\ 8.6319 & 3.8416 & 6.9223 & 6.9223 & 3.8416 & 8.6319 \\ 8.6319 & 3.8416 & 6.9223 & 6.9223 & 3.8416 & 8.6319 \\ 8.9880 & 11.2079 & 4.9880 & 4.9880 & 11.2079 & 8.9880 \\ 8.9880 & 11.2079 & 4.9880 & 4.9880 & 11.2079 & 8.9880 \end{bmatrix}$$

$$\bar{\rho}=53.4072 \qquad \bar{Q}=[21.9904\ \ \underline{22.4303}\ \ 20.9635\ \ 20.9635\ \ \underline{22.4303}\ \ 21.9904]$$

A.3 The controllability measures of Hamdan and Nayfeh

$$F=\begin{bmatrix} 0.9369 \\ 0.9369 \\ 0.0856 \\ 0.0856 \\ 0.1119 \\ 0.1119 \\ 0.1676 \\ 0.1676 \\ 0.2864 \\ 0.2864 \\ 0.5569 \\ 0.5569 \end{bmatrix} \quad \Psi^T B=\begin{bmatrix} 0.2173 & 0.3916 & 0.4883 & 0.4883 & 0.3916 & 0.2173 \\ 0.2173 & 0.3916 & 0.4883 & 0.4883 & 0.3916 & 0.2173 \\ 0.0199 & 0.0358 & 0.0446 & 0.0446 & 0.0358 & 0.0199 \\ 0.0199 & 0.0358 & 0.0446 & 0.0446 & 0.0358 & 0.0199 \\ 0.0468 & 0.0583 & 0.0260 & 0.0260 & 0.0583 & 0.0468 \\ 0.0468 & 0.0583 & 0.0260 & 0.0260 & 0.0583 & 0.0468 \\ 0.0873 & 0.0389 & 0.0700 & 0.0700 & 0.0389 & 0.0873 \\ 0.0873 & 0.0389 & 0.0700 & 0.0700 & 0.0389 & 0.0873 \\ 0.1492 & 0.0664 & 0.1197 & 0.1197 & 0.0664 & 0.1492 \\ 0.1492 & 0.0664 & 0.1197 & 0.1197 & 0.0664 & 0.1492 \\ 0.2327 & 0.2902 & 0.1292 & 0.1292 & 0.2902 & 0.2327 \\ 0.2327 & 0.2902 & 0.1292 & 0.1292 & 0.2902 & 0.2327 \end{bmatrix}$$

$$Q=[0.5174 \;\; 0.7045 \;\; \underline{0.7443} \;\; \underline{0.7443} \;\; 0.7045 \;\; 0.5174]$$

A.4 The observability measures of Hamdan and Nayfeh

$$P=\begin{bmatrix} \underline{1.3162} \\ 1.2263 \\ 1.2025 \\ 1.2025 \\ 1.2263 \\ \underline{1.3162} \end{bmatrix} \quad C\Phi=\begin{bmatrix} 0.0811 & 0.0811 & 0.2311 & 0.2311 & 0.4153 & 0.4153 & 0.5138 & 0.5138 & 0.4993 & 0.4993 & 0.3471 & 0.3471 \\ 0.1461 & 0.1461 & 0.4164 & 0.4164 & 0.5178 & 0.5178 & 0.2286 & 0.2286 & 0.2222 & 0.2222 & 0.4328 & 0.4328 \\ 0.1821 & 0.1821 & 0.5192 & 0.5192 & 0.2305 & 0.2305 & 0.4120 & 0.4120 & 0.4004 & 0.4004 & 0.1926 & 0.1926 \\ 0.1821 & 0.1821 & 0.5192 & 0.5192 & 0.2305 & 0.2305 & 0.4120 & 0.4120 & 0.4004 & 0.4004 & 0.1926 & 0.1926 \\ 0.1461 & 0.1461 & 0.4164 & 0.4164 & 0.5178 & 0.5178 & 0.2286 & 0.2286 & 0.2222 & 0.2222 & 0.4328 & 0.4328 \\ 0.0811 & 0.0811 & 0.2311 & 0.2311 & 0.4153 & 0.4153 & 0.5138 & 0.5138 & 0.4993 & 0.4993 & 0.3471 & 0.3471 \end{bmatrix}$$

$$G=[0.3495 \;\; 0.3495 \;\; 0.9963 \;\; 0.9963 \;\; 0.9937 \;\; 0.9937 \;\; 0.9859 \;\; 0.9859 \;\; 0.9581 \;\; 0.9581 \;\; 0.8306 \;\; 0.8306]$$

Acknowledgment

The authors would like to thank the Ministry of Science and Technology of Korea for financial support in the form of grant(M1-0203-00-0017-02J0000-00910) under the NRL(National Research Laboratory).

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